

# SOLUTION OF THE DIRAC EQUATION WITH MOBIUS SQUARE POTENTIAL PLUS RING SHAPED POTENTIAL UNDER THE FRAMEWORK OF SPIN SYMMETRY VIA SHAPE INVARIANCE

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**Abstract** – In this paper we investigate Dirac equation with spin symmetry for Mobius square potential plus ring-shaped potential under the framework of supersymmetric quantum mechanics (SUSYQM). We construct symmetrical partner Hamiltonians which satisfy shape invariance and via a one-to-one mapping, the relativistic energy eigen value is obtained. We solve for the wave functions from the hypergeometric differential equations which are developed by transforming the radial and polar equations via SUSYQM.

**Keywords**–Dirac equation, Mobius square potential, Shape invariance, Spin symmetry.

## 1 INTRODUCTION

Schrodinger equation is the workhorse equation of non-relativistic quantum mechanics. Klein Gordon equation is suitable for the description of relativistic spinless particles like pions, mesons,  $\pi^\pm$ ,  $\pi^0$ .

First order Duffin – Kemmer – petiau equation is best for the description of relativistic spin 0 and spin 1 bosons.

Dirac equation is ideal for the description of relativistic spin  $\frac{1}{2}$  particles like electrons. Dirac equation is invariant with respect to Unitary transformation, gauge transformation, Lorentz transformation, the transformation matrix  $s$ , space inversion, charge conjugation and time reversal.

Dirac equation has facilitated the understanding of the phenomenon of negative energy states. In recent times, research has pivoted on quantum mechanical wave equations because of the significance of the solutions in quantum mechanics and other fields. The exact solutions of central and non-central potentials find applications in molecular physics, nuclear physics, quantum chemistry, etc. The exact solutions of Dirac equation for ring – shaped potentials can be used for the description of ring shaped

molecules like benzene and the interactions between the deformed pair of nuclei<sup>1</sup>[37].

According to Dirac theory, spin symmetry arises when the difference between the repulsive Lorentz vector potential  $V(r)$  and the attractive Lorentz Scalar potential  $S(r)$ , is a constant (i.e.  $\Delta(r) = V(r) - S(r) = \text{constant} = C_s$ ). Pseudospin arises when the sum is a constant (i.e.  $\Sigma(r) = V(r) + S(r) = \text{constant} = C_{ps}$ ).

Implicitly, for spin symmetry,  $\frac{d(\Delta r)}{dr} = 0$  and  $\Sigma(r)$  is the potential under investigation and for pseudospin symmetry,  $\frac{d\Sigma(r)}{dr} = 0$  and  $\Delta r$  is the potential under investigation.

Pseudospin symmetry was observed more than four decades ago in spherical atomic nuclei and was later introduced in nuclear physics to take care of the experimentally observed

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quasi-degeneracy in single nucleon doublets between normal parity shell model orbitals  $(n, l, j=l+1/2)$  and  $(n-1, l+2, j=l+3/2)$ , where,  $n, l$ , and  $j$  represent the radial, angular and total angular momentum quantum numbers respectively [16], [17], [18], [19].

Symmetry limits of  $(D+1)$  – dimensional Dirac equation with mobius square potential has already been investigated. [9].

Also, approximate k-state solutions to the Dirac Mobius square – Yukawa and Mobius square – quasi Yukawa problems under pseudospin and spin symmetry limits with Coulomb – like Tensor interaction has already been investigated [10].

IN ADDITION, relativistic spin and pseudospin symmetries of inversely, quadratic Yukawa – like plus Mobius square potentials including a Coulomb – like Tensor interaction has already been investigated [11].

Furthermore, Pseudospin symmetry of Dirac equation for a Mobius square plus Mie type potential with a Coulomb – like Tensor interaction via SUSYQM has already been investigated [47].

Dirac equation with different potentials under the framework of spin and pseudospin symmetries using various techniques like Supersymmetric Quantum Mechanics (SUSYQM), Nikiforov – Uvarov method (NU), Asymptotic Iteration Method (AIM), Kustaanheimo-Stiefel method (KS), Group theoretical method, Factorization method, Path integral method, Darboux transformation, Tridiagonal representation, etc, have been investigated by many authors.

In this paper, we intend to solve Dirac equation for generalized Mobius square plus ring – shaped potential under the framework of spin symmetry using SUSYQM.

This paper is organized as follows. In section 2, we give a brief introduction of Dirac theory and the decoupling of the generalized Dirac equation. In section 3, we briefly give an introduction of SUSYQM formalism. Section 4 is purposely for the solutions of the decoupled equations using SUSYQM. A brief conclusion is given section 5.

## 2 DIRAC EQUATION WITHOUT TENSOR COUPLING

The most general Dirac equation for spin  $1/2$  particles moving in an attractive scalar potential  $S(r)$ , a repulsive vector potential  $V(r)$  and a tensor  $U(r)$  in the relativistic unit is, [22].

$$\left[ \vec{\alpha} \cdot \vec{p} + \beta (M + S(r)) - i \vec{\beta} \cdot \vec{\alpha} \hat{r} U(r) \right] \psi(r) = [E - V(r)] \psi(r) \quad (1)$$

where  $E$  is the relativistic energy of the system,  $\vec{p} = -i\vec{\nabla}$  is the three dimensional operator and  $M$  the mass of the particle,  $\alpha$  and  $\beta$  are  $4 \times 4$  Pauli matrices.

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \quad (2)$$

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (3)$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli matrices satisfy the following anti-commutation relations

$$\hat{\alpha}_i \cdot \hat{\alpha}_j + \hat{\alpha}_j \cdot \hat{\alpha}_i = 2\delta_{ij} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad (4)$$

$$\hat{\alpha}_j \cdot \hat{\beta} + \hat{\beta} \cdot \hat{\alpha}_j = 0 \quad (5)$$

$$\hat{\alpha}_j^2 = \hat{\beta}^2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad (6)$$

$$P_j = -i \frac{\partial}{\partial x_j} \quad (1 \leq j \leq 3) \quad (7)$$

Dirac equation with scalar and vector potential without tensor coupling in the relativistic unit (i.e.  $\hbar = c = 1$ ) is given by

$$\nabla^2 + (E_{nk} - M - \sum (r\Delta)(f)M + E_{nk} - V) F(r, \theta, \phi) = 0 \quad (8)$$

where the laplacian operator in spherical coordinates is given by

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \quad (9)$$

In spherical coordinates, the most general potential  $Z(r, \theta, \phi)$  for which Dirac equation is separable is given by

$$Z(r, \theta) = Z(r) + \frac{Z(\theta)}{r^2} \quad (10)$$

$$Z(r, \theta) = V_0 + V_1 \left( \frac{A + Be^{-ar}}{C + De^{-ar}} \right) + V_2 \left( \frac{A + Be^{-ar}}{C + De^{-ar}} \right)^2 + \left( \frac{\beta + \rho \cos \theta}{r^2 \sin \theta} \right) \quad (11)$$

For this situation, the wave function  $F_{nk}(r, \theta, \phi)$  is given by

$$F_{nk}(r, \theta, \phi) = \frac{R(r)}{r} \frac{H(\theta)}{\sqrt{\sin \theta}} \psi(\phi) \quad (12)$$

Applying the standard procedure of separation of variables, the decoupled equations for the functions  $R(r), H(\theta)$  and  $\psi(\phi)$  are obtained and they are as follows:

$$-\frac{d^2 R(r)}{dr^2} + \left( \gamma Z(r) + \frac{\lambda}{r^2} \right) R(r) = \varepsilon^2 R(r) \quad (13)$$

$$-\frac{d^2 H(\theta)}{d\theta^2} + \left( \gamma Z(\theta) + m_l^2 \operatorname{cosec}^2 \theta - \frac{\operatorname{cosec}^2 \theta}{4} - \frac{1}{4} \right) H(\theta) = \lambda H(\theta) \quad (14)$$

$$-\frac{d^2 \psi(\phi)}{d\phi^2} + \gamma Z(\phi) = m_l^2 \psi(\phi) \quad (15)$$

where  $\lambda$  and  $m_l^2$  are separation constants and

$$\gamma = (M + E_{nk} - C_s), \quad \varepsilon^2 = (E_{nk} - M)(M + E_{nk} - C_s)$$

### 3 SUPERSYMMETRIC QUANTUM MECHANICS

In supersymmetric quantum mechanics formalism, the hamiltonian can be expressed in terms of creation ( $A^+$ ) and annihilation ( $A^-$ ) operators. Also, it can be expressed in a factorized form given by

$$H^- = A^+ A^- \quad \text{and} \quad H^+ = A^- A^+ \quad (16)$$

The creation and annihilation operators are defined by

$$A^- = -\frac{\hbar}{\sqrt{2m}} \frac{\partial}{\partial x} + \phi(x), \quad A^+ = \frac{\hbar}{\sqrt{2m}} \frac{\partial}{\partial x} + \phi(x) \quad (17)$$

Where  $\phi(x)$  is the superpotential

$$H^- \psi^-(x, a) = E^- \psi^-(x, a) \quad (18)$$

$$A^+ A^- \psi^-(x, a) = E^- \psi^-(x, a) \quad (19)$$

$$\left( \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + \phi^2(x) - \frac{\hbar}{\sqrt{2m}} \phi'(x) \right) \psi^-(x, a) = E^- \psi^-(x, a) \quad (20)$$

$$\left( \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + V_-(x) \right) \psi^-(x, a) = E^- \psi^-(x, a) \quad (21)$$

Comparing (20) and (21)

$$V_-(x) = \phi^2(x) - \frac{\hbar}{\sqrt{2m}} \phi'(x) \quad (22)$$

$$H^+ \psi^+(x, a) = E^+ \psi^+(x, a) \quad (23)$$

$$A^- A^+ \psi^+(x, a) = E^+ \psi^+(x, a) \quad (24)$$

$$\left( \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + \phi^2(x, a) + \frac{\hbar}{\sqrt{2m}} \phi'(x) \right) \psi^+(x, a) = E^+ \psi^+(x, a) \quad (25)$$

$$\left( \frac{-\hbar^2}{2m} \frac{d^2}{dx^2} + V_+(x) \right) \psi^+(x, a) = E^+ \psi^+(x, a) \quad (26)$$

Comparing (25) and (26)

$$V_+(x) = \phi^2(x) + \frac{\hbar}{\sqrt{2m}} \phi'(x) \quad (27)$$

In supersymmetric quantum mechanics, it is customary to describe  $V_-(x)$  (i.e. one of the partner Hamiltonians) in terms of its ground state wave function  $\Psi_0^{(-)}$  ( $\equiv \Psi_0^-$ ), whose corresponding ground state energy  $E_0^-$  is adjusted to zero (unbroken symmetry).

$$H^- \bar{\Psi}(x) = A^+ \left( A^- \bar{\Psi}(x) \right) = 0 \quad (28)$$

From (28)

$$\psi_0^-(x) = N \exp \left( -\frac{\sqrt{2m}}{\hbar} \int_{x_0}^x \phi(y) dy \right) \quad (29)$$

Where N is the normalization constant

In this formalism, solvability is guaranteed when the Hamiltonians satisfy the shape invariance relation given by

$$V_+(a_n, x) = V_-(a_{n+1}, x) + R(a_{n+1}) \quad (30)$$

Where  $a_{n+1}$  is a new set of parameters determined from the old set via a one-to-one mapping  $F: a_n \mapsto a_{n+1} = F(a_n)$  and  $R(a_n)$  is an additive constant which is independent of the variable  $x$

When shape invariance is satisfied, the energy spectrum of the  $n^{\text{th}}$  Hamiltonian is determined via

$$E_n^- = \sum_{k=1}^n R(a_k) \quad (31)$$

The total energy of the system is given by

$$E_{n,m,l}^- = E_n^- + E_{0,m,l}^- + E_0^- \quad (32)$$

Where  $E_{0,m,l}^-$  is the energy due to broken symmetry and  $E_0^-$  is the energy due to unbroken symmetry, which is usually adjusted to zero

Also when shape invariance is satisfied, the wave function is obtained from

$$\psi_{n-1}^{(+)} = \frac{\hbar}{\sqrt{E_n^-}} A^- \psi_n^{(+)}, \quad \psi_n^{(-)} = \frac{\hbar}{\sqrt{E_{n-1}^+}} A^+ \psi_{n-1}^{(-)} \quad (33)$$

#### 4 SOLUTION OF DIRAC EQUATION

In this section, we intend to solve Dirac equation with Mobius square potential plus ring shaped potential under the framework of spin symmetry via shape invariance using SUSYQM. According to Dirac theory, spin symmetry arises when  $\Delta(r) = V(r) - S(r) = \text{Constant} = C_s$  or  $\frac{d(\Delta(r))}{dr} = 0$ ,

while  $\sum(r)$  is the potential under investigation. The potential under study is given by

$$Z(r, \theta) = V_0 + V_1 \left( \frac{A + Be^{-ar}}{C + De^{-ar}} \right) + V_2 \left( \frac{A + Be^{-ar}}{C + De^{-ar}} \right)^2 + \left( \frac{\beta + \rho \cos \theta}{r^2 \sin \theta} \right) \quad (34)$$

Where,  $V_0, V_1, A, B, C, D, \alpha$  are constant coefficients.  $\beta$  and  $\rho$  are dimensionless real quantum parameters related to the angle dependent potentials, while  $r$  and  $\theta$  are the radial and polar coordinates respectively.

$$Z(r, \theta) = Z(r) + Z(\theta) \quad (35)$$

$$Z(r) = V_0 + V_1 \left( \frac{A + Be^{-ar}}{C + De^{-ar}} \right) + V_2 \left( \frac{A + Be^{-ar}}{C + De^{-ar}} \right)^2 \quad (36)$$

$$Z(\theta) = \frac{\beta + \rho \cos \theta}{\sin^2 \theta} \quad (37)$$

Equation (36) is the radial potential, while equation (37) is the angle dependent (i.e. polar) potential

#### 4.1 Solution of The Radial Equation

$$-\frac{dR(r)}{dr^2} + V_{\text{eff}}(r)R(r) = E_{n,m,l}^- R(r) \quad (38)$$

Comparing equation (13) and equation (38),

$\mathcal{E}^2 = E_{n,m,l}^-$  is the total relativistic eigen energy value and  $V_{\text{eff}}(r)$  is the effective potential

$$V_{\text{eff}}(r) = \gamma \left( V_0 + V_1 \left( \frac{s + te^{-ar}}{1 - \omega e^{-ar}} \right) + V_2 \left( \frac{s + te^{-ar}}{1 - \omega e^{-ar}} \right)^2 \right) + \frac{\lambda}{r^2} \quad (39)$$

Where  $\frac{A}{C} = s, \frac{B}{C} = t, \frac{D}{C} = -\omega$

Introducing

$$\frac{1}{r^2} \approx \frac{\alpha^2 e^{-ar}}{(1 - e^{-ar})^2} \quad (40)$$

as the approximation of the centrifugal term, since (13) is not exactly solvable [48].

Substituting (40) in (39), we obtain

$$V_{\text{eff}}(r) = \frac{1}{(1 - \omega e^{-ar})^2} (\beta_1 e^{-2ar} + \beta_2 e^{-ar} + \beta_3) \quad (41)$$

Where,

$$\beta_1 = (\gamma V_2 t^2 - \gamma V_1 t \omega) \quad (42)$$

$$\beta_2 = (\gamma V_1 t + 2\gamma V_2 s t + \lambda \alpha^2 - \gamma V_1 s \omega) \quad (43)$$

$$\beta_3 = (\gamma V_0 + \gamma V_1 s + \gamma V_2 s^2) \quad (44)$$

Assuming that the superpotential is of the form

$$\phi(r) = Q_1 + \frac{Q_2 e^{-\alpha r}}{(1 - \omega e^{-\alpha r})^2} \quad (45)$$

Therefore,

$$(\phi(r))^2 = Q_1^2 + \frac{Q_2^2 e^{-2\alpha r}}{(1 - \omega e^{-\alpha r})^2} + \frac{2Q_1 Q_2 e^{-\alpha r}}{(1 - \omega e^{-\alpha r})} \quad (46)$$

$$\phi'(r) = -\frac{\alpha Q_2 e^{-\alpha r}}{(1 - \omega e^{-\alpha r})^2} \quad (47)$$

And considering that the superpotential is the solution of the RICCATI equations

$$V_{eff}^-(r) = (\phi(r))^2 - \phi'(r) + E_{0,m,l}^- \quad (48)$$

$$V_{eff}^+(r) = (\phi(r))^2 + \phi'(r) + E_{0,m,l}^- \quad (49)$$

Substituting (46) and (47) in (48)

$$V_{eff}^-(r) = \frac{1}{(1 - \omega e^{-\alpha r})^2} \{ (Q_2^2 + \omega^2 E_{0,m,l}^- + \omega^2 Q_1^2 - 2Q_1 Q_2 \omega) e^{-2\alpha r} + (\alpha Q_2 + 2Q_1 Q_2 - 2\omega E_{0,m,l}^- - 2\omega Q_1^2) e^{-\alpha r} + Q_1^2 + E_{0,m,l}^- \} \quad (50)$$

Comparing (41) and (50),

$$\beta_1 = Q_2^2 + \omega^2 E_{0,m,l}^- + \omega^2 Q_1^2 - 2Q_1 Q_2 \omega \quad (51)$$

$$\beta_2 = \alpha Q_2 + 2Q_1 Q_2 - 2\omega E_{0,m,l}^- - 2\omega Q_1^2 \quad (52)$$

$$\beta_3 = Q_1^2 + E_{0,m,l}^- \quad (53)$$

$$(51) + \omega(52) + \omega^2(53),$$

$$\beta_1 + \omega\beta_2 + \omega^2\beta_3 = \alpha\omega Q_2 + Q_2^2 \text{ which implies}$$

$$Q_2^2 + (\alpha\omega)Q_2 - (\beta_1 + \omega\beta_2 + \omega^2\beta_3) = 0 \quad (54)$$

Substituting (53) in (51),

$$Q_1 = \left( \frac{Q_2^2 + \omega^2\beta_3 - \beta_1}{2Q_2\omega} \right) \quad (55)$$

From (54),

$$Q_2 = \frac{-\alpha\omega}{2} \left\{ 1 \pm \sqrt{1 + \frac{4(\beta_1 + \omega\beta_2 + \omega^2\beta_3)}{\alpha^2\omega^2}} \right\} \quad (56)$$

#### 4.1.1 Shape Invariance for Radial Coordinate

$$V_-(r, a_0) = (\phi(r))^2 - \phi'(r) \quad (57)$$

$$V_-(r, a_0) = Q_1^2 + \frac{Q_2(Q_2 + \alpha)e^{-\alpha r}}{(1 - \omega e^{-\alpha r})^2} + \frac{2Q_1 Q_2 e^{-\alpha r}}{(1 - \omega e^{-\alpha r})} + \frac{Q_2^2(e^{-2\alpha r} - e^{-\alpha r})}{(1 - \omega e^{-\alpha r})^2} \quad (58)$$

$$V_+(r, a_0) = (\phi(r))^2 + \phi'(r) \quad (59)$$

$$V_+(r, a_0) = Q_1^2 + \frac{Q_2(Q_2 - \alpha)e^{-\alpha r}}{(1 - \omega e^{-\alpha r})^2} + \frac{2Q_1 Q_2 e^{-\alpha r}}{(1 - \omega e^{-\alpha r})} + \frac{Q_2^2(e^{-2\alpha r} - e^{-\alpha r})}{(1 - \omega e^{-\alpha r})^2} \quad (60)$$

Considering the mapping

$$Q_2 \mapsto (Q_2 + \alpha) \text{ which implies } a_n = Q_2 + n\alpha \text{ and } a_0 = Q_2$$

The eigen energy spectrum equation corresponding to the supersymmetric potential  $V_-$  (energy due to shape invariance) is given by

$$E_n^- = \sum_{k=1}^n R(a_k) = R(a_1) + R(a_2) + R(a_3) + \dots + R(a_n) \quad (61)$$

For solvability, the partner Hamiltonians satisfy the shape invariance relation

$$R(a_1) = V_+(r, a_0) - V_-(r, a_1) \quad (62)$$

$$R(a_1) = \left( \frac{a_0^2 + \omega^2\beta_3 - \beta_1}{2a_0\omega} \right)^2 - \left( \frac{a_1^2 + \omega^2\beta_3 - \beta_1}{2a_1\omega} \right)^2 \quad (63)$$

$$R(a_2) = \left( \frac{a_1^2 + \omega^2\beta_3 - \beta_1}{2a_1\omega} \right)^2 - \left( \frac{a_2^2 + \omega^2\beta_3 - \beta_1}{2a_2\omega} \right)^2 \quad (64)$$

$$R(a_3) = \left( \frac{a_2^2 + \omega^2\beta_3 - \beta_1}{2a_2\omega} \right)^2 - \left( \frac{a_3^2 + \omega^2\beta_3 - \beta_1}{2a_3\omega} \right)^2 \quad (65)$$

⋮

$$R(a_n) = \left( \frac{a_{n-1}^2 + \omega^2\beta_3 - \beta_1}{2a_{n-1}\omega} \right)^2 - \left( \frac{a_n^2 + \omega^2\beta_3 - \beta_1}{2a_n\omega} \right)^2 \quad (66)$$

$$E_n^- = \left( \frac{a_0^2 + \omega^2 \beta_3 - \beta_1}{2a_0 \omega} \right)^2 - \left( \frac{a_n^2 + \omega^2 \beta_3 - \beta_1}{2a_n \omega} \right)^2 \quad (67)$$

(67) is the eigen energy value due to shape invariance.

From (53),

$$E_{0,m,l}^- = \beta_3 - Q_1^2 = \beta_3 - \left( \frac{a_0^2 + \omega^2 \beta_3 - \beta_1}{2a_0 \omega} \right)^2 \quad (68)$$

(68) is the energy due to broken symmetry.

$E_0$  is the energy due to unbroken symmetry.

In supersymmetric quantum mechanics formalism, the ground state energy of one of the Hamiltonians ( $E_0$ ) is usually adjusted to zero for unbroken symmetry.

The total energy spectrum is given by

$$E_{n,m,l}^- = \varepsilon^2 = E_n^- + E_{0,m,l}^- + E_0 \quad (69)$$

$$E_{n,m,l}^- = \beta_3 - \frac{1}{4} \left( a_n + \frac{(\beta_3 - \beta_1)}{a_n} \right)^2 \quad (70)$$

where

$$a_n = \alpha(n + \sigma) \quad (71)$$

$$\sigma = -\frac{1}{2} \left( 1 \pm \sqrt{1 + \frac{4}{\alpha^2} (\beta_1 + \beta_2 + \beta_3)} \right) \quad (72)$$

The total relativistic eigen energy spectrum equation is given by

$$(E_{nk} - M)(M + E_{nk} - C_s) = \beta_3 - \frac{1}{4} \left( a_n + \frac{\beta_3 - \beta_1}{a_n} \right)^2 \quad (73)$$

The radial wave function is the solution of the hypergeometric differential equation which is obtained by substituting the transformation  $y = \omega e^{-\alpha r}$  in (13). The transformed equation is given by

$$\frac{d^2 R(y)}{dy^2} + \frac{(1-y)}{y(1-y)} \frac{dR(y)}{dy} + \frac{[\Omega_1 y^2 + \Omega_2 y + \Omega_3]}{y^2(1-y)^2} R(y) = 0 \quad (74)$$

Where

$$\Omega_1 = \frac{E_{n,m,l}^- - (\beta_1 / \omega^2)}{\alpha^2 \omega^2} \quad (75)$$

$$\Omega_2 = \frac{-2E_{n,m,l}^- + (\beta_2 / \omega)}{\alpha^2 \omega^2} \quad (76)$$

$$\Omega_3 = \frac{E_{n,m,l}^- + \beta_3}{\alpha^2 \omega^2} \quad (77)$$

The solution of (74) is a hypergeometric function and it is given by

$$R_{nl}(r) = N_{nl} y^{\frac{(\mu-1)}{2}} (1-y)^{\frac{(1+\nu)}{2}} {}_2F_1(-n, n + \mu + \nu; \mu; y) \quad (78)$$

Where  $N_{nl}$  is the normalization constant,

$$\frac{\mu-1}{2} = \sqrt{-\Omega_3}, \quad \frac{(1+\nu)}{2} = \left( -\Omega_1 - \Omega_2 - \Omega_3 + \frac{1}{4} \right)^{\frac{1}{2}} + \frac{1}{2}. \quad (79)$$

Thus,

$$R_{nl}(r) = N_{nl} (\omega e^{-\alpha r})^{\sqrt{-\Omega_3}} (1 - \omega e^{-\alpha r})^{\frac{1}{2} + \sqrt{-\Omega_1 - \Omega_2 - \Omega_3 + \frac{1}{4}}} x \quad (80)$$

$${}_2F_1 \left( -n, n + 2\sqrt{-\Omega_3} + 1 + 2\sqrt{-\Omega_1 - \Omega_2 - \Omega_3 + \frac{1}{4}}; 2\sqrt{-\Omega_3} + 1; \omega e^{-\alpha r} \right)$$

According to the normalization condition  $\int_0^{\infty} |R(r)|^2 = 1$ , it

implies that

$$N_{nl}^2 \int_0^1 y^{\frac{(\mu-1)}{2}} (1-y)^{\frac{(1+\nu)}{2}} {}_2F_1(-n, n + \mu + \nu; \mu; y) dy = 1 \quad (81)$$

Using the condition of orthogonality of Jacobi polynomials

$$\int_0^1 y^{\frac{(\mu-1)}{2}} (1-y)^{\frac{(1+\nu)}{2}} {}_2F_1(-n, n + \mu + \nu; \mu; y) dy$$

$$= \frac{n!}{\mu + \nu + 2n} \times \frac{\Gamma^2(\mu) \Gamma(\mu + 1 + \nu)}{\Gamma(\mu + \nu + n) \Gamma(\mu + \nu)} \quad (82)$$

Therefore,

$$N_{nl} = \left[ \frac{\mu + \nu + 2n}{n!} \times \frac{\Gamma(\mu + \nu + n) \Gamma(\mu + \nu)}{\Gamma^2(\mu) \Gamma(\mu + 1 + \nu)} \right]^{1/2} \quad (83)$$

The complete radial wave function is given by

$$R_{nl}(r) = \left[ \frac{\mu + \nu + 2n}{n!} x \frac{\Gamma(\mu + \nu + n) \Gamma(\mu + \nu)}{\Gamma^2(\mu) \Gamma(\nu + 1 + n)} \right]^{1/2} x \left[ (\omega e^{-ar})^{\sqrt{-\Omega_1}} (1 - \omega e^{-ar})^{\frac{1}{2} \sqrt{-\Omega_1 - \Omega_2 - \Omega_3 + \frac{1}{4}}} \right] x \left[ {}_2F_1 \left( -n; n + 2\sqrt{-\Omega_3} + 1 + 2 \left( -\Omega_1 - \Omega_2 - \Omega_3 + \frac{1}{4} \right)^{\frac{1}{2}}; 2\sqrt{-\Omega_3} + 1; \omega e^{-ar} \right) \right] \quad (84)$$

#### 4.2 Solution of the Polar Equation

$$-\frac{d^2 H(\theta)}{d\theta^2} + V_{eff}(\theta) H(\theta) = \lambda H(\theta) \quad (85)$$

Comparing (14) and (85)

$$V_{eff}(\theta) = (\gamma\beta + \gamma\delta \cos \theta + m_l^2 - (1/4)) \operatorname{cosec}^2 \theta - (1/4) \quad (86)$$

Assuming that the superpotential is of the form

$$\phi(\theta) = p \tan \theta - q \cot \theta \quad (87)$$

Therefore,

$$(\phi(\theta))^2 = p^2 \sec^2 \theta + q^2 \operatorname{cosec}^2 \theta - (p+q)^2 \quad (88)$$

$$\phi'(\theta) = p \sec^2 \theta + q \operatorname{cosec}^2 \theta \quad (89)$$

The superpotential is considered to be the solution of the RICCATI equations

$$V_{eff}^-(\theta) = (\phi(\theta))^2 - \phi'(\theta) + E_{0,m,l}^- \quad (90)$$

$$V_{eff}^+(\theta) = (\phi(\theta))^2 + \phi'(\theta) + E_{0,m,l}^- \quad (91)$$

Substituting (88) and (89) in (90),

$$V_{eff}^-(\theta) = p(p-1) \sec^2 \theta + q(q-1) \operatorname{cosec}^2 \theta - (p+q)^2 + E_{0,m,l}^- \quad (92)$$

Comparing (86) and (92)

$$q^2 - q = (\gamma\beta + \gamma\delta \cos \theta + m_l^2 - (1/4)) \quad (93)$$

$$p^2 - p = 0 \quad (94)$$

$$-\frac{1}{4} = -(p+q)^2 + E_{0,m,l}^- \quad (95)$$

From (93)

$$q = \frac{1}{2} \left\{ 1 \pm \sqrt{1 + 4 \left( \gamma\beta + \gamma\delta \cos \theta + m_l^2 - \frac{1}{4} \right)} \right\} \quad (96)$$

From (94)

$$p = 1 \quad (97)$$

From (95)

$$E_{0,m,l}^- = (p+q)^2 - (1/4) \quad (98)$$

(98) is the energy due to unbroken symmetry for the polar coordinate

#### 4.2.1 Shape Invariance for the Polar Coordinate

Super symmetric partner potentials are given by

$$V_-(\theta, a_0) = (\phi(\theta))^2 - \phi'(\theta) \quad (99)$$

$$V_-(\theta, a_0) = p(p-1) \sec^2 \theta + q(q-1) \operatorname{cosec}^2 \theta - (p+q)^2 \quad (100)$$

$$V_+(\theta, a_0) = (\phi(\theta))^2 + \phi'(\theta) \quad (101)$$

$$V_+(\theta, a_0) = p(p+1) \sec^2 \theta + q(q+1) \operatorname{cosec}^2 \theta - (p+q)^2 \quad (102)$$

The eigen energy equation due to shape invariance is given by

$$E_n^- = \sum_{k=1}^n R(a_k) = R(a_1) + R(a_2) + R(a_3) + \dots + R(a_n) \quad (103)$$

For solvability, the partner Hamiltonians satisfy the shape invariance relation

$$R(a_1) = V_+(\theta, a_0) - V_-(\theta, a_0) \quad (104)$$

$$R(p+1, q+1) = -[(p+0) + (q+0)] + [(p+1) + (q+1)] \quad (105)$$

$$R(p+2, q+2) = -(p+q+2) + (p+q+4) \quad (106)$$

$$R(p+3, q+3) = -(p+q+4) + (p+q+6) \quad (107)$$

⋮

$$R(p+n, q+1) = -(p+(n-1) + q+(n-1)) + (p+n+q+n) \quad (108)$$

$$E_n^- = -(p+q)^2 + (p+q+2n)^2 \quad (109)$$

(109) is the eigen energy value due to shape invariance.

$$E_{0,m,l}^- = (p+q)^2 - (1/4) \quad (110)$$

(110) is the energy due to broken symmetry.

The total energy spectrum is given by

$$E_{n,m,l}^- = E_n^- + E_{0,m,l}^- + E_0^- \quad (111)$$

$$E_{n,m,l}^- = -(p+q)^2 + (p+q+2n)^2 + (p+q)^2 - (1/4) + 0 \quad (112)$$

$$E_{n,m,l}^- = (p+q+2n)^2 - (1/4) \quad (113)$$

$$\lambda = l^2 = (p+q+2n)^2 - (1/4) \quad (114)$$

$$E_{n,m,l}^- = \lambda \quad (115)$$

(115) is the total energy spectrum.

The polar wave function is of the form  $H(\theta) = (\sin \theta)^q (\cos \theta)^p Y(\theta)$ , where  $Y(\theta)$  is the spherical harmonics and it is obtained by introducing a new variable  $x = \sin^2 \theta$  in equation (14), transforming it to a hypergeometric differential equation given by

$$x(1-x) \frac{d^2 y}{dx^2} + \left\{ \left( q + \frac{1}{2} \right) - \left[ \left( q + \frac{1}{2} \right) + \left( p + \frac{1}{2} \right) \right] \right\} x \left\{ x \frac{dy}{dx} + n(p+q+n)y(x) \right\} = 0 \quad (116)$$

Thus,

$$H(\theta) = N_n (\sin \theta)^q (\cos \theta)^p F(-n, p+q+n; q; x) \quad (117)$$

Where  $N_n$  is the normalization constant and  $F(-n, p+q+n; q; x)$  is a hypergeometric function

According to the normalization condition  $\int_0^H H(\theta)^2 d\theta = 1$

where

$$H(\theta) = N_n Z^{\eta-1} (1-Z)^{t-\eta} {}_2F_1(-n, t+n; \eta; Z) \quad (118)$$

$$H(\theta) = N_n Z^{\eta-1} (1-Z)^{t-\eta} {}_2F_1(-n, t+n; \eta; Z), \quad (119)$$

$$\eta = q + \frac{1}{2} \quad t = p + q, \quad (120)$$

it implies that

$$N_n^2 \int_0^1 Z^{\eta-1} (1-Z)^{t-\eta} {}_2F_1^2(-n, t+n; \eta; Z) dZ = 1 \quad (121)$$

Using the condition of orthogonality of Jacobi polynomials

$$\int_0^1 Z^{\eta-1} (1-Z)^{t-\eta} {}_2F_1^2(-n, t+n; \eta; Z) dZ = \frac{n!}{(t+2n)!} x \frac{\Gamma^2(\eta) \Gamma(t+n-\eta+1)}{\Gamma(t+n) \Gamma(\eta+n)} \quad (122)$$

$$N_n = \left[ \frac{(t+2n)!}{n!} x \frac{\Gamma(t+n) \Gamma(\eta+n)}{\Gamma^2(\eta) \Gamma(t+n-\eta+1)} \right]^{1/2} \quad (123)$$

The complete polar wave function is given by

$$H(\theta) = \left[ \frac{(p+q+2n)!}{n!} x \frac{\Gamma(p+q+n) \Gamma(q+\frac{1}{2}+n)}{\Gamma^2(q+\frac{1}{2}) \Gamma(p+\frac{1}{2}+n)} \right]^{1/2} (\sin \theta)^q (\cos \theta)^p F(-n; p+q+n; q; \sin^2 \theta) \quad (124)$$

The standard solution of the azimuthal equation is given by

$$\psi(\phi) = \frac{1}{\sqrt{2\pi}} e^{im_l \phi} \quad (125)$$

The total wave function is

$$F_{mm}(r, \theta, \phi) = \left\{ \left[ \frac{\mu+\nu+2n}{n!} x \frac{\Gamma(\mu+\nu+n) \Gamma(\mu+\nu)}{\Gamma^2(\mu) \Gamma(\nu+1+n)} \right]^{1/2} \left[ \frac{(\omega e^{-i\theta})^{\sqrt{-\Omega_3}} (1-\omega e^{-i\theta})^{\frac{1}{2} \sqrt{-\Omega_3 - \Omega_3 + \frac{1}{4}}}}{r} \right] x \right\} \left[ {}_2F_1 \left( -n; n+2\sqrt{-\Omega_3}+1+2 \left( -\Omega_1 - \Omega_2 - \Omega_3 + \frac{1}{4} \right)^{\frac{1}{2}}; 2\sqrt{-\Omega_3}+1; \omega e^{-i\theta} \right) \right] \left\{ \left[ \frac{(p+q+2n)!}{n!} x \frac{\Gamma(p+q+n) \Gamma(q+\frac{1}{2}+n)}{\Gamma^2(q+\frac{1}{2}) \Gamma(p+\frac{1}{2}+n)} \right]^{1/2} \left( \frac{(\sin \theta)^q (\cos \theta)^p F(-n; p+q+n; q; \sin^2 \theta)}{\sqrt{\sin \theta}} \right) x \right\} \frac{e^{im_l \phi}}{\sqrt{2\pi}} \quad (126)$$

## 5 CONCLUSION

In this work, we have shown that using SUSYQM, one can easily obtain the eigen value without solving the Schrödinger-like differential equation. Also, we have demonstrated that using SUSYQM, a hyper geometric differential equation can be established, from which the

normalized wave functions can easily be determined in terms of Jacobi polynomials. Furthermore, we have shown that Dirac equation with certain potentials that cannot be solved using the traditional method can be solved using SUSYQM.

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